Série 5b Solutions

Exercise 5b.1 – Power shaft

The electric motor in Figure 5b.1 exerts a torque of 800 N·m at point A on the steel shaft ABCD when it is rotating at a constant speed. The point D of the shaft is free to rotate. Design specifications require that the diameter of the shaft be uniform from A to D and that the angle of twist between A and D not exceed 1.5°. Knowing that $\tau_{max} \leq 60$ MPa and $G = 77$ GPa.

Determine the minimum diameter shaft that can be used.

Hint: Consider both maximum stress and maximum twist criterion for this analysis.

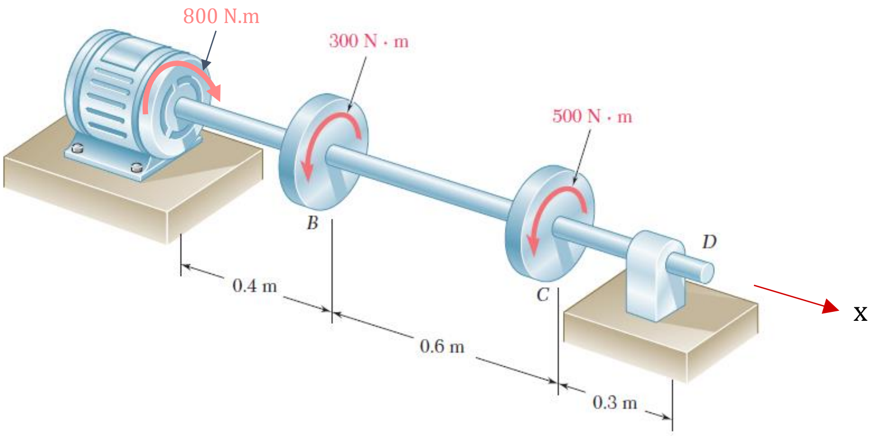


Figure 5b.1 | Power shaft system description.

Solution – 5b.1

What is Given:

Motor torque: -800 N·m (*value is negative to offset the torques from B and C)
$\tau_{max} \leq 60$ MPa
Maximum twist angle: 1.5°
$G = 77$ GPa

Find Minimum diameter shaft that can be used: In order to find the minimum radius for the power shaft we will consider both the given twist and stress criterion and then select the smallest allowed diameter.

Relevant Methods/Equations:

Maximum torsion equation

$$\tau_{\max} = \frac{Tc}{I_p} \tag{5b.1.1}$$

Twist angle equation

$$\phi = \frac{TL}{GI_p} \tag{5b.1.2}$$

Polar area moment of inertia for a cylinder

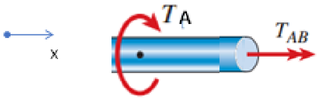
$$I_p = \frac{\pi r^4}{2} \tag{5b.1.3}$$

Find torques for each segment of the shaft:

We use the equilibrium equation on the whole system to find the Torque in A, with $T_A = -800\ \mathrm{N \cdot m}$

We use the equilibrium equation, $\Sigma M_z = 0$, and the method of sections to find the different torque for

$$T_{CD} + T_D = 0 \text{ with } T_D = 0 \text{ we find } T_{CD} = 0 \tag{5b.1.4}$$



$$T_{AB} + T_A = 0 \text{ with } T_A = -800\ \mathrm{N \cdot m} \text{ we find } T_{AB} = 800\ \mathrm{N \cdot m} \tag{5b.1.5}$$

$$T_{BC} + T_B + T_A = 0 \text{ with } T_B = 300\ \mathrm{N \cdot m} \text{ and } T_A = -800 \text{ we find } T_{BC} = 500\ \mathrm{N \cdot m} \tag{5b.1.6}$$

Design based on stress:

$$\tau_{max} = \frac{Tc}{I_p} = \frac{2T}{\pi r^3} \tag{5b.1.7}$$

$$r = \sqrt[3]{\frac{2T}{\pi \tau_{\max}}} = \sqrt[3]{\frac{2 \cdot 800}{\pi (60 \cdot 10^6)}} = 20.4 \cdot 10^{-3}\ \mathrm{m} \tag{5b.1.8}$$

Design based on deformation:

$$\phi_{max} = 1.5° = 26.18 \cdot 10^{-3}\ \mathrm{rad}$$

$$\phi_{D/C} = 0 \tag{5b.1.9}$$

$$\phi_{C/B} = \frac{T_{BC} L_{BC}}{GI_p} = \frac{(500 * 0.6)}{GI_p} = \frac{300}{GI_p} \tag{5b.1.10}$$

$$\phi_{B/A} = \frac{T_{AB} L_{AB}}{GI_p} = \frac{(800 * 0.4)}{GI_p} = \frac{320}{GI_p} \tag{5b.1.11}$$

$$\phi_{D/A} = \phi_{D/C} + \phi_{C/B} + \phi_{B/A} = \frac{620}{G\left(\frac{\pi}{2}\right) r^4} = \frac{2 * 620}{\pi * G * r^4} \tag{5b.1.12}$$

$$r = \sqrt[4]{\frac{(2)(620)}{\pi G \phi_{max}}} = \sqrt[4]{\frac{(2)(620)}{\pi (77 \cdot 10^9)(26.18 \cdot 10^{-3})}} = 21.04 \cdot 10^{-3}\ \mathrm{m} \tag{5b.1.13}$$

With the deformation criterion requiring the largest radius we can conclude that **minimum allowable diameter is 42.1 mm.**

Exercise 5b.2 – Non-cylindrical bars

Shafts A and B in Figure 5b.2 are made of the same material and have the same cross-sectional area, but A has a circular cross section and B has a square cross section. Assume both deformations to be elastic.

a) Determine the ratio of the maximum torques, T_A and T_B, that can be safely applied to A and B, respectively.

b) Determine the ratio of the maximum values of the angles, ϕ_A and ϕ_B, through which shafts A and B, respectively, may be twisted.

Hint: For a square member, $C_1 = 0.208$ and $C_2 = 0.1406$

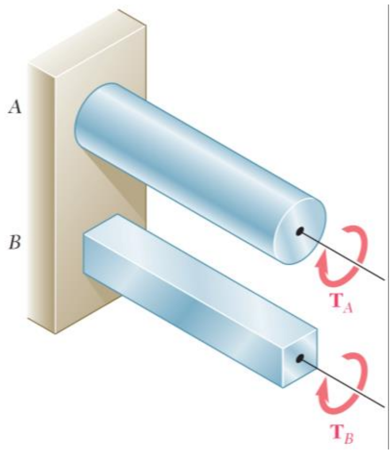


Figure 5b.2 | Bars with the same cross-sectional area with (A) Circular and (B) Square cross-sections.

Solution – 5b.2

Let *r* = Radius of circular section A, and *b* = Side Length of square section B

So

$$Area_A = \pi r^2$$

$$Area_B = b^2$$

For equal areas:

$$\pi r^2 = b^2 \tag{5b.2.1}$$

$$r = \frac{b}{\sqrt{\pi}} \tag{5b.2.2}$$

a) Determine the ratio of the maximum torques, T_A and T_B

For circular section A:

$$\tau_A = \frac{T_A r}{I_P} = \frac{2T_A}{\pi r^3} \tag{5b.2.3}$$

$$T_A = \left(\frac{\pi}{2}\right) r^3 \tau_A \tag{5b.2.4}$$

For Square section B:

$$C_1 = 0.208 \tag{5b.2.5}$$

$$\tau_B = \frac{T_B}{C_1 ab^2} = \frac{T_B}{C_1 b^3} \tag{5b.2.6}$$

$$T_B = \tau_B C_1 b^3 \tag{5b.2.7}$$

The ratio would then be:

$$\frac{T_A}{T_B} = \frac{\left(\frac{\pi}{2}\right) r^3 \tau_A}{\tau_B C_1 b^3} = \frac{\left(\frac{\pi}{2}\right)\left(\frac{b}{\sqrt{\pi}}\right)^3 \tau_A}{\tau_B C_1 b^3} = \frac{1}{2C_1\sqrt{\pi}}\left(\frac{\tau_A}{\tau_B}\right) \tag{5b.2.8}$$

If the stresses are the same, $\tau_A = \tau_B$ then:

$$\frac{T_A}{T_B} = \frac{1}{2C_1\sqrt{\pi}}\left(\frac{\tau_A}{\tau_B}\right) = \frac{1}{2C_1\sqrt{\pi}} \approx 1.356 \tag{5b.2.9}$$

b) Determine the ratio of the maximum values of the angles, ϕ_A and ϕ_B

For the Circular Section A:

$$\gamma_{MAX} = \frac{\tau_A}{G} = \frac{r\phi_A}{L} \tag{5b.2.10}$$

$$\phi_A = \frac{L\tau_A}{rG} \tag{5b.2.11}$$

For Square Section B:

$$\tau_B = \frac{T_B}{C_1 b^3} = \frac{T_B}{(0.208)b^3} \tag{5b.2.12}$$

$$T_B = 0.208 b^3 \tau_B \tag{5b.2.13}$$

$$\phi_B = \frac{T_B L}{C_2 ab^3 G} = \frac{0.208 * b^3 \tau_B L}{0.1406 * b^4 G} = \frac{1.4794 * L\tau_B}{bG} \tag{5b.2.14}$$

Making the ratio:

$$\frac{\phi_A}{\phi_B} = \frac{\left(\frac{L\tau_A}{rG}\right)}{\left(\frac{1.4794 * L\tau_B}{bG}\right)} = 0.676 * \frac{b\tau_A}{r\tau_B} \approx 0.676\sqrt{\pi}\left(\frac{\tau_A}{\tau_B}\right) \tag{5b.2.15}$$

For equal stresses, $\tau_A = \tau_B$ then:

$$\frac{\phi_A}{\phi_B} = 0.676\sqrt{\pi} \approx 1.198 \tag{5b.2.16}$$

Exercise 5b.3 – From material to device

We want to integrate a new composite material in an inertial sensor combining accelerometer and gyroscope together. We consider a cylinder rod (radius $r = 2$ [μm], length $L = 4$ [mm]) as the spin-axis of a gyroscope with a steel mass $m = 8\pi * 10^{-6}$ [kg] at its free end. A torque $T_0 = 48\pi$ [μN · μm] is applied at the interface between the steel mass and the end of the cylinder rod. The shear modulus is $G = 40$ [GPa].

The mass of the rod is neglected and the gravitation constant is $g = 10 \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]$.

a) Give an expression and calculate the values for the normal stresses along x, y, z of the cylinder rod at point A.

b) Give an expression and calculate the value of the maximum shear stress due to the applied torque at point A as a function of *r* and T_0.

c) Give an expression for the torsion angle of the cylinder rod at point A of the bar and calculate its value.

d) Calculate the maximum shear stress in the bar of the structure.

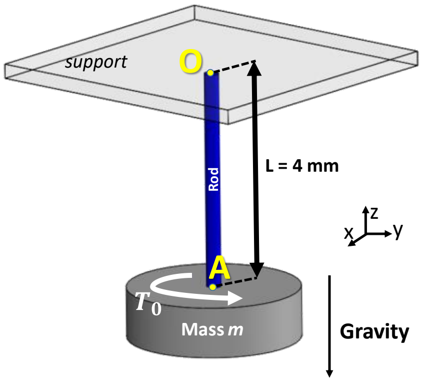


Figure 5b.3 | Inertial sensor combining accelerometer and gyroscope.

Solution – 5b.3

a) Give an expression and calculate the values for the normal stresses along x, y, z of the cylinder rod at point A.

$$\sigma_x = 0, \quad \sigma_y = 0 \tag{5b.3.1}$$

$$\sigma_z = \frac{m * g}{A} = \frac{(8\pi * 10^{-6}\ [\mathrm{kg}])\left(10\left[\frac{\mathrm{m}}{s^2}\right]\right)}{\pi(2 * 10^{-6}[m])^2} = 20\ [\mathrm{MPa}] \tag{5b.3.2}$$

b) Give an expression and calculate the value of the maximum shear stress due to the applied torque at point A as a function of *r* and T_0.

$$\tau_{yz} = \frac{\mathrm{T_{int}r}}{\mathrm{I_p}} \tag{5b.3.3}$$

$$T_{int} = -\mathrm{T_0} \tag{5b.3.4}$$

$$I_p = \frac{\pi r^4}{2} \tag{5b.3.5}$$

$$\tau_{yz,max} = -\frac{2T_0}{\pi r^3} = -12\text{ MPa} \tag{5b.3.6}$$

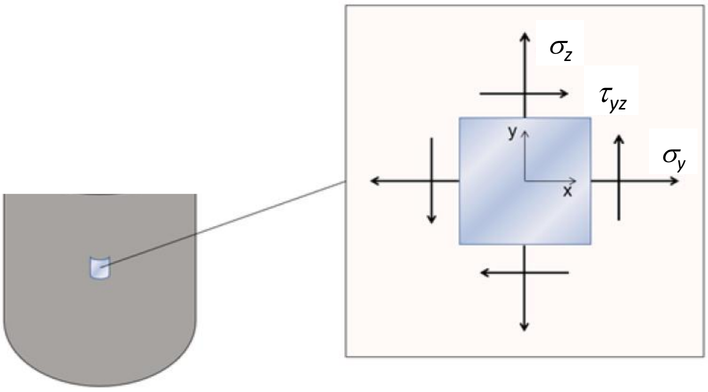
c) Give an expression for the torsion angle of the cylinder rod at point A of the bar and calculate its value.

$$\phi = -\frac{2T_0 L}{\pi G r^4} = -0.6\text{ rad} = -34.4° \tag{5b.3.7}$$

d) Calculate the maximum shear stress in the bar of the structure.

$$\sigma_z = 20\text{ [MPa] and } \tau_{yz} = -12\text{ MPa} \tag{5b.3.8}$$

In general (not taking into account the signs of the stresses), the stress state at the surface the bar can be represented using a 2D element in the plane y,z as shown below.



The maximum shear stress is given by a combination of the normal and shear stresses applying this relationship:

$$\tau_{\substack{Max\\min}} = \pm\sqrt{\left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{yz}^2} \tag{5b.3.9}$$

$$\tau_{\substack{Max\\min}} = \pm\sqrt{\left(\frac{0-20}{2}\right)^2 + 12^2} = \pm\sqrt{100+144} = \pm 15.6\text{ [MPa]} \tag{5b.3.10}$$

Exercise 5b.4 – MEMS Torque Magnetometer

A "Torque magnetometer" is a MEMS device used to measure the magnetic properties of a material with great sensitivity. These devices typically consist of a cantilever beam with a small amount of test material attached to the unclamped end. An external magnetic field is used to exert a torque on the sample, twisting the end of the beam. This deflection is then measured and, by repeating this with different conditions, the magnetic properties of the sample can be empirically determined with great sensitivity.

The magnetometer can be modelled simply with a cylindrical beam where the test material attached to the end as shown in Figure 5b.4. A uniform magnetic field is introduced to the sample with field strength $\boldsymbol{B}$**,** which we assume is applied at point R. We assume the two materials are firmly attached and have the same cross section. We also assume the beam material is completely non-magnetic with $G_{cantilever} = 50$ GPa, while the shear modulus of the sample material is $G_{Sample} = 25$ GPa.

Recall that the magnetic torque on the beam is:

$$\boldsymbol{T}_{\boldsymbol{mag}} = \boldsymbol{m} \times \boldsymbol{B}$$

Where $\boldsymbol{B}$ is the magnetic field and $\boldsymbol{m}$ is the magnetic moment of the sample.

For a uniform magnetic field $B = 20$ T, we measure a twist angle at the end of the sample of 2 degrees. **What is the magnetic moment of the sample?**

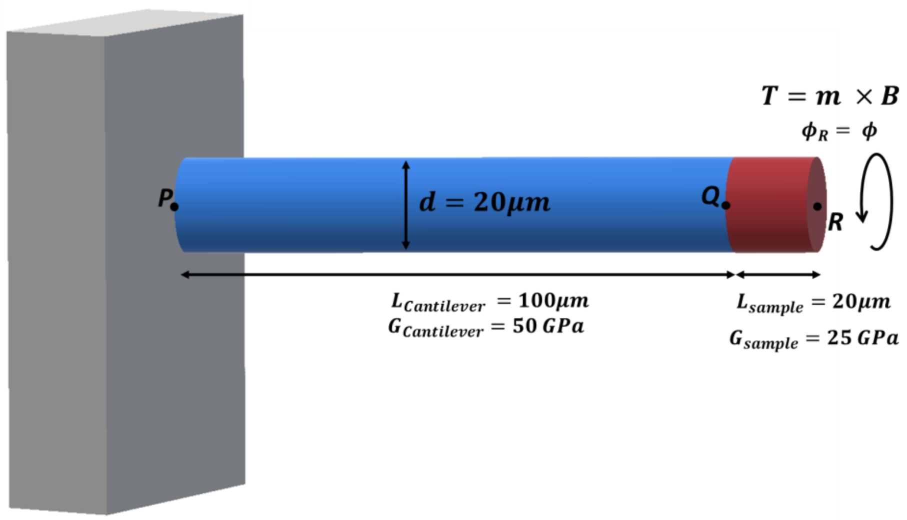


Figure 5b.4 | Simplified model of a MEMS Torque Magnetometer as a cylinder.

Solution – 5b.4

Find the magnetic moment, *m*:

You have first to consider the polar moment of inertia, I_p. It is only depending on the geometry of the bar. We call d the diameter of the bar.

$$I_p = \frac{\pi d^4}{32} \tag{5b.4.1}$$

Then, we calculate the separate stiffness, *k,* of the two portions of the bar. Calling L the length of the bar, it gives us for each portion:

$$k_x = G_x \frac{I_p}{L_x} \tag{5b.4.2}$$

The equivalent stiffness of serial portions is the indirect sum of the respective stiffness of each portion of the bar. Therefore:

$$k_{eq} = \left(k_{PQ}^{-1} + k_{QR}^{-1}\right)^{-1} \tag{5b.4.3}$$

$$k_{PQ} = G_{PQ} \frac{I_p}{L_{PQ}} = 7.85 \cdot 10^{-6}$$

$$k_{QR} = G_{QR} \frac{I_p}{L_{QR}} = 1.96 \cdot 10^{-5}$$

$$k_{eq} = 5.61 \cdot 10^{-6}$$

To conclude, we obtain the torque, T, with respect to the torsion angle, ϕ:

$$T = k_{eq} * \phi \tag{5b.4.4}$$

$$T = (5.61 \cdot 10^{-6}) * \left(2deg * \frac{\pi}{180}\right) = 1.958 \cdot 10^{-7}\ N \cdot \mathrm{m}$$

Substituting, the numerical application gives:

$$\begin{gathered} T = m \times B \\ m = T/B \\ m = 9.79 \cdot 10^{-9}\ N \cdot \mathrm{m/T} \end{gathered} \tag{5b.4.5}$$

Exercise 5b.5 – Torsion of a coated tube

We consider a circular hollow tube coated with a hard material, as shown in Figure 5b.5. A Torque, T, is applied at its right end side at distance L. The tube is composed of a Silicon Nitride (SiN) coating and a Steel inner part. The material properties are: for the Silicon Nitride coating $E_{SiN} = 175$ GPa and $\nu_{SiN} = 0.25$, and for the Steel part $E_{Steel} = 208$ GPa and $\nu_{Steel} = 0.30$. The inner radius, R_1, is 0.03 m, the thickness of the Steel part ($t_{Steel} = R_2 - R_1 = 90$ mm), the thickness of the Silicon Nitride coating ($t_{SiN} = R_3 - R_2 = 2$ mm) and the length of the tube is $L = 1$ m.

- **a) Calculate the numerical value of the shear moduli for both parts (Steel and SiN).**
- **b) Calculate the numerical value of the polar moments of inertia for both parts (Steel and SiN).**
- **c) The yield shear stresses are $\tau_{yield,Steel} = 220$ MPa and $\tau_{yield,SiN} = 410$ MPa. With a safety factor (SF) of 2, calculate the angle of twist, ϕ, in radians (rad), for both materials. Indicate which part of the coated tube fails first.**
- **d) We want now to deform the whole tube with an angle of twist, $\phi = 0.0275$ rad. Considering a safety factor (SF) of 2, and that the inner radius, R_1, and SiN thickness remain the same:**
  - **i. Calculate the value of the new maximum thickness of the Steel tube before failure occurs.**
  - **ii. For the new dimensions of the coated tube, plot the graph of torsional shear stress as function of the radius, r, for $x = L$. Indicate in your plot the stress values at the inner and outer edges of the Steel and SiN parts.**

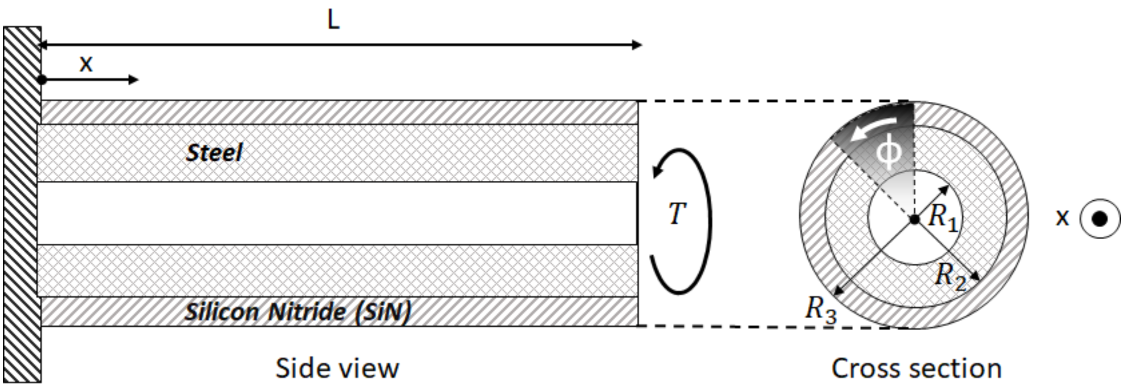


Figure 5b.5 | Schematic of the coated tube (The drawing is not to scale).

Solution – 5b.5

a) Determine the shear moduli

Shear modulus, Steel

$$G_{Steel} = \frac{E_{Steel}}{2(1 + \nu_{Steel})} = \frac{208 \cdot 10^9}{2(1 + 0.3)} = 80 \text{ GPa}$$

Shear modulus, Silicon nitride (SiN)

$$G_{SiN} = \frac{E_{SiN}}{2(1 + \nu_{SiN})} = \frac{175 \cdot 10^9}{2(1 + 0.25)} = 70 \text{ GPa}$$

b) Determine the polar moments

$$I_{Steel} = \frac{\pi\left({R_2}^4 - {R_1}^4\right)}{2} = 3.2445 \cdot 10^{-4}\ \ \mathrm{m}^4$$

$$I_{SiN} = \frac{\pi\left({R_3}^4 - {R_2}^4\right)}{2} = 0.2226 \cdot 10^{-4}\ \ \mathrm{m}^4$$

c) Calculate the angle of twist in radians (rad) at which the composite fails

Necessary formulation

$$\tau(r) = \mathrm{G}\gamma(r)$$

$$\gamma(r) = \frac{\phi r}{\mathrm{L}}$$

$$\phi_{failure} = \frac{\tau_{\max}\mathrm{L}}{2\mathrm{G_n R_n}}$$

Angular of twist of Steel part at failure

$$\phi_{Steel} = \frac{1}{SF} * \frac{\tau_{Steel}L}{G_{Steel}R_2} = 0.5 * \frac{220 \cdot 10^6}{80 \cdot 10^9 * 0.12} = 0.0115\ \mathrm{rad}$$

Angle of twist of SiN part at failure

$$\phi_{SiN} = \frac{1}{SF} * \frac{\tau_{SiN}L}{G_{SiN}R_3} = 0.5 * \frac{410 \cdot 10^6}{70 \cdot 10^9 * 0.122} = 0.0240\ \mathrm{rad}$$

The max angle before the failure of the coated tube is in the steel tube at an angle of 0.0115 rad.

d) Under an angle of twist, ϕ, of 0.0275 rad. safety factor (SF) of 2, and fixed inner radius (R_1) and thickness of Silicone Nitride (t_{SiN}). Find the following:

i. The new thickness of the Steel tube to avoid failure

The thickness of the steel tube has to be adapted in order to allow the structure to reach the desired angular deformation.

Previous formula can be used but now the angle of twist has to be fixed and the only variable will be the radius R'_2.

$$R'_2 = \frac{1}{SF}\frac{\tau_{Steel}L}{G_{Steel}\phi} = 0.5 * \frac{220 \cdot 10^6}{80 \cdot 10^9 * 0.0275} = 0.05\ \mathrm{m}$$

So the thickness of steel tube to avoid failure at an angle of 0.0275 rad is equal to

$$t_{Steel} = R'_2 - R_1 = 0.05 - 0.03 = 0.02\ \mathrm{m}$$

ii. The stress values and sketch of stress profile

With the new dimensions known, we can now calculate the new stress values at the interfaces. The safety factor can now be ignored, as we're looking at the actual stress profiles in the bar.

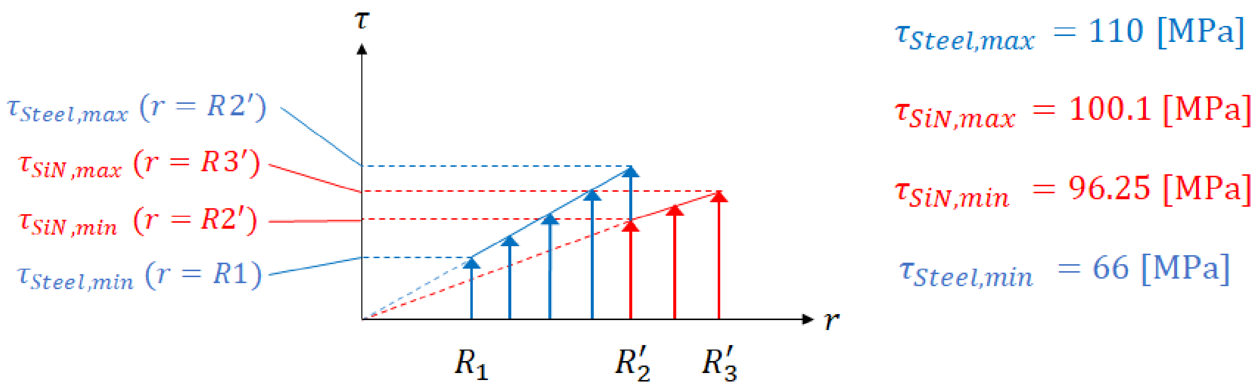
$$\tau_{Steel}(R_1) = \frac{\phi_{Steel} * G_{Steel} * R_1}{L} = \frac{0.0275 * 80 \cdot 10^9 * 0.03}{1} = 66.0\ \mathrm{MPa}$$

$$\tau_{SiN}(R'_2) = \frac{\phi_{SiN} * G_{SiN} * R'_2}{L} = \frac{0.0275 * 70 \cdot 10^9 * 0.05}{1} = 96.25\ \mathrm{MPa}$$

$$\tau_{SiN}(R'_3) = \frac{\phi_{SiN} * G_{SiN} * R'_3}{L} = \frac{0.0275 * 70 \cdot 10^9 * 0.052}{1} = 100.1\ \mathrm{MPa}$$

The value of the stress at R_2 was given as 220 MPa, but since we overestimated by a factor two, the actual stress at this point will be 110 MPa

When we draw the stress profile it looks like the figure below



Two parts are important here.

- First, there is **an offset in the stress between the less rigid and more rigid materials at $\mathbf{R_2'}$.** As well as an offset of the stress at R_1, due to the hole.
- Secondly, **the slope of the shear in the SiN material is less steep than that of the Steel part**, as it has a smaller modulus of rigidity, G, than Steel. So under the same strain, it will not undergo the same amount of stress, since we are still in the linear regime and it behaves Hookean. ($G_{Steel} = 80$ GPa, $G_{SiN} = 70$ GPa)